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Linearized Kappa Guidance

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An alternative solution to the κ-guidance dynamic equations is presented in this paper. The κ-guidance is a midcourse guidance algorithm that maximizes intercept velocity. The nonlinear κ-guidance dynamic equations are linearized. The linearization is via a nonlinear coordinate transformation and nonlinear feedback. The cost function is approximated by a quadratic cost. The linear quadratic regulator (LQR) method is then used to solve the optimal control problem. A closed-form solution to the LQR problem is derived for both the free and fixed final angle of approach cases. A closed-form solution has not been derived for the original solution. A simulation is conducted to compare the trajectories and costs of the linearized solution versus the original solution. The results show similar performance between the two methods.

Nomenclature

R = range to predicted intercept point
γ = velocity vector angle
γ_f = terminal velocity vector angle constraint
δ = heading error angle
κ = curvature of trajectory
ω = coefficient depending on aerodynamic parameters, a constant

I. Introduction

In this paper, an alternate solution method to the κ-guidance optimal control problem is presented. The κ-guidance is a technique for computing the midcourse guidance of a missile, e.g., a surface-to-air missile. The κ-guidance optimizes the midcourse trajectory of a missile so that the terminal velocity is maximized. This increases the probability of kill. The current solution method for the κ-guidance optimal control problem (which we will also refer to as the original method) is to write the Hamiltonian and obtain a solution from the necessary conditions. (In this report the Hamiltonian is in the optimal control context.) Closed-form feedback controllers are obtained for both the free final angle of approach case and the fixed final angle of approach case in the original method. The original solution is suboptimal as approximations are made in its derivation. In this alternate method, the nonlinear dynamics, which also vary with respect to the independent parameter, range, are linearized via a nonlinear coordinate transformation and nonlinear feedback. The cost function, which is not quadratic, is still not quadratic in the transformed coordinates. A quadratic approximation to the cost function is made in the transformed coordinates. Then the (transformed) guidance problem is solved using linear quadratic regulator (LQR) control on the linearized dynamics and approximate quadratic cost. Closed-form state equations and adjoint variable equations are obtained. That is, the velocity vector angle and heading error angle for the midcourse are known in closed form with the alternate solution. Because of the cost approximation, the linearized solution is also suboptimal. The main feature of the linearized solution presented here is that closed-form solutions for the states are derived. Such closed-form solutions for the states were not obtained in the original method. The costs of trajectories computed by each method for identical initial conditions are comparable.

In Sec. II, the κ-guidance optimal control problem is reviewed. A detailed discussion of κ-guidance in its original form is given in Lin.[1] In Sec. III the dynamics of the problem are linearized using a nonlinear coordinate transformation and nonlinear feedback. The LQR optimal control problem is set up in Sec. IV. In Sec. V, the necessary conditions are given and solutions considered for free and fixed final angles of approach. Examples comparing the original and alternate solutions are given in Sec. VI. Much of this paper was presented in Serakos and Lin.[2]

II. Dynamic Equations

In this section, the dynamic (state) equations and optimal control problem will be reviewed. The κ-guidance dynamic equations are

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}R} &= -\kappa\sec(\delta)\\ \frac{\mathrm{d}\delta}{\mathrm{d}R} &= -\kappa\sec(\delta) - \frac{\tan(\delta)}{R}\end{aligned} \tag{1}$$

The independent parameter in this problem is taken to be R. It is interesting to note that the derivative of R depends on δ, as $\dot{R}$ is positive or negative depending on whether $\frac{1}{2}\pi < \delta < \frac{3}{2}\pi$ or $-\frac{1}{2}\pi < \delta < \frac{1}{2}\pi$. Under every normal circumstance, however, $-\frac{1}{2}\pi < \delta < \frac{1}{2}\pi$. The control is κ. The cost is

$$\mathcal{J} = \int_0^R \left(\tfrac{1}{2}\kappa^2 + \omega^2\right)\sec(\delta)\,\mathrm{d}R \tag{2}$$

This original formulation was developed by Reifler.[3] Figure 1 shows the relationships between the states. The state equations (1) may be derived from the geometry and the definition of κ. The argument of Eq. (2) contains κ^2, which penalizes excessive curvature, and $\sec(\delta)$, which penalizes large heading error angle. These work to maximize intercept velocity, which increases the probability of kill. The constant ω is defined by (8-150) of Lin.[1]

We rewrite the dynamic equations to be in a form that is more familiar to control engineers. Let $x_1 = \gamma$ and $x_2 = \delta$. Then, Eq. (1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}R}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\-\tan(x_2)/R\end{bmatrix} - \sec(x_2)\begin{bmatrix}1\\1\end{bmatrix}\kappa \tag{3}$$

The variables x_1 and x_2 are referred to as the state variables. Let

$$f(x_1, x_2, R) = \begin{bmatrix}0\\-\tan(x_2)/R\end{bmatrix}, \quad g(x_1, x_2, R) = -\sec(x_2)\begin{bmatrix}1\\1\end{bmatrix} \tag{4}$$

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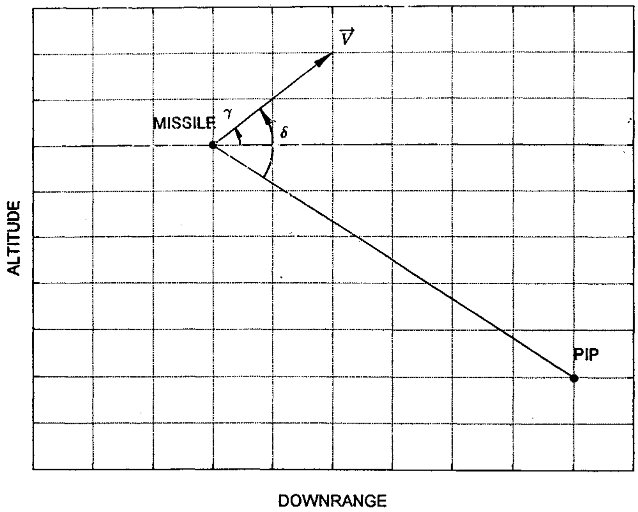

Fig. 1 Midcourse guidance problem.

Then Eq. (3) becomes

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}R} = f(x_1, x_2, R) + g(x_1, x_2, R)\kappa \tag{5}$$

Equations (5) and (2) represent the κ-midcourse guidance optimal control problem. The original solution proceeded to solve this problem by obtaining the Hamiltonian and solving for the necessary conditions. A detailed discussion of the original solution of κ-guidance is given in Sec. 8.6.3 of Lin.[1] The state equations are given in Lin[1] by (8-140a, b), and the cost is given by (8-150). Also in Ref. 1 the closed-form solution for the feedback control is given by (8-167) in the free final angle of approach case and by (8-165) when the final angle of approach is specified. The original solution is suboptimal since approximations are made in obtaining these controllers; see Sec. 8.6.3 of Lin.[1]

III. Linearizing Coordinate Transformation and Feedback

In this section, we will find a nonlinear coordinate transformation and feedback that will linearize the state equation (5). Consider the coordinate transformation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = F(x_1, x_2, R) = \begin{bmatrix} -x_1 + x_2 \\ -\tan(x_2)/R \end{bmatrix} \tag{6}$$

Note that the transformation exists for all $R \neq 0$ and $-\frac{1}{2}\pi < x_2 < \frac{1}{2}\pi$. The inverse transformation is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = F^{-1}(z_1, z_2, R) = \begin{bmatrix} -z_1 - \tan^{-1}(Rz_2) \\ -\tan^{-1}(Rz_2) \end{bmatrix} \tag{7}$$

Define a nonlinear feedback by

$$\kappa = \alpha(x_1, x_2, R) + \beta(x_1, x_2, R)u \tag{8}$$

where

$$\alpha(x_1, x_2, R) = -\frac{\sin(x_2)\left[\cos^2(x_2) + 1\right]}{R} \tag{9}$$

$$\beta(x_1, x_2, R) = R\cos^3(x_2) \tag{10}$$

and u is the input to the linearized closed-loop system. Using the coordinate transformation and nonlinear feedback, we compute the representation of the closed-loop system in z coordinates. Denote

$$\frac{\partial F}{\partial \boldsymbol{x}} = \begin{bmatrix} -1 & 1 \\ 0 & -\dfrac{\sec^2(x_2)}{R} \end{bmatrix} \doteq F_* \tag{11}$$

The closed-loop system [Eq. (5) with nonlinear feedback (8)] is

$$\frac{\mathrm{d}z}{\mathrm{d}R} = F_*(f + g\alpha)(\boldsymbol{x}) + F_*(g\beta)(\boldsymbol{x})u + \frac{\partial F}{\partial R} \tag{12}$$

where

$$F_*(f + g\alpha) = F_* \begin{bmatrix} 0 \\ -\dfrac{\tan(x_2)}{R} \end{bmatrix} + F_*[-\sec(x_2)]\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{-\sin(x_2)\left[\cos^2(x_2) + 1\right]}{R}$$

$$= \begin{bmatrix} -\tan(x_2)/R \\ -\tan(x_2)/R^2 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_2/R \end{bmatrix}$$

$$\frac{\partial F}{\partial R} = -\begin{bmatrix} 0 \\ \dfrac{z_2}{R} \end{bmatrix}$$

$$F_*(g\beta) = F_*[-\sec(x_2)]\begin{bmatrix} 1 \\ 1 \end{bmatrix}R\cos^3(x_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence, the closed-loop system in z coordinates is

$$\frac{\mathrm{d}}{\mathrm{d}R}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \tag{13}$$

It is interesting to note that the right-hand side of Eq. (12) depends on R, whereas the right-hand side of Eq. (13) does not. Apparently, the coordinate transformation F and nonlinear feedback α and β vary with respect to R in such a way that the dependence on R in Eq. (13) cancels out.

IV. LQR Optimal Control Problem Formulation

In this section we formulate the LQR optimal control problem. The cost function (2) is not in the desired quadratic form after applying the nonlinear feedback (8) and coordinate transformation (6). Hence, a quadratic approximation to the cost will be made. Along with the linearized dynamics (13), this completes the formulation of the LQR optimal control problem.

We make the following quadratic approximations [see Eqs. (9) and (10)]:

$$\alpha \approx -\frac{x_2\left[\left(1 - \frac{1}{2}x_2^2\right)^2 + 1\right]}{R} \approx -\frac{x_2\left(2 - x_2^2\right)}{R} \tag{14}$$

$$\beta \approx R\left(1 - \tfrac{1}{2}x_2^2\right)^3 \approx R\left(1 - \tfrac{3}{2}x_2^2\right) \tag{15}$$

Hence, for small x_2 and u

$$\frac{1}{2}\kappa^2 = \frac{1}{2}\alpha^2 + \alpha\beta u + \frac{1}{2}\beta^2 u^2 \approx \frac{2x_2^2}{R^2} - 2x_2 u + \frac{1}{2}R^2u^2 \tag{16}$$

For small x_2, the secant may be approximated by

$$\sec(x_2) = \frac{1}{\cos(x_2)} \approx \frac{1}{1 - \frac{1}{2}x_2^2}\,\frac{1 + \frac{1}{2}x_2^2}{1 + \frac{1}{2}x_2^2} \approx 1 + \frac{1}{2}x_2^2 \tag{17}$$

Since this is a quadratic approximation, x_2 and u do not have to be as small as they would have to be if a linear approximation were being made. This point will be further considered in the examples. From Eqs. (16) and (17),

$$\left(\frac{1}{2}\kappa^2 + \omega^2\right)\sec(x_2) \approx \left(\frac{2}{R^2} + \frac{1}{2}\omega^2\right)x_2^2 - 2x_2u + \frac{1}{2}R^2u^2 + \omega^2 \tag{18}$$

The $+\omega^2$ term on the right-hand side of Eq. (18) may be deleted without changing the optimal control problem; hence, the cost may be approximated by

$$\mathcal{J} \approx \int_0^R \left[\left(\frac{2}{R^2} + \frac{1}{2}\omega^2\right)x_2^2 - 2x_2u + \frac{1}{2}R^2u^2\right]\mathrm{d}R \tag{19}$$

The coordinate transformation and its inverse may similarly be approximated [Eqs. (6) and (7)]:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \approx \begin{bmatrix} -x_1 + x_2 \\ -x_2/R \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} -z_1 - z_2R \\ -z_2R \end{bmatrix} \tag{20}$$

In z coordinates, the cost may be approximated by Eqs. (19) and (20):

$$\mathcal{J} \approx \int_0^R \left[\left(2 + \frac{1}{2}R^2\omega^2\right)z_2^2 + 2Rz_2u + \frac{1}{2}R^2u^2\right]\mathrm{d}R \tag{21}$$

The LQR optimal control problem is given by Eqs. (13) and (21). The solution to this LQR optimal control problem would be an approximate solution to the original optimal control problem given by Eqs. (1) and (2).

V. Necessary Conditions

In this section, the necessary conditions, and their solution, for both the free and fixed final approach angle cases will be considered.

A. Formulation of Hamiltonian and Optimal Controller

The Hamiltonian is (see Ref. 4)

$$\mathcal{H}(z, p, u, R) \doteq \mathcal{L}(z, u, R) + \langle p, f(z, u)\rangle \tag{22}$$

where

$$\mathcal{L}(z, u, R) \doteq \left(2 + \tfrac{1}{2}R^2\omega^2\right)z_2^2 + 2Rz_2u + \tfrac{1}{2}R^2u^2 \tag{23}$$

$$f(z, u) \doteq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u = \begin{bmatrix} z_2 \\ u \end{bmatrix} \tag{24}$$

The vector $p = (p_1, p_2)'$ contains the adjoint variables. Substituting Eqs. (23) and (24) into Eq. (22) gives the Hamiltonian

$$\mathcal{H} = \left(2 + \tfrac{1}{2}R^2\omega^2\right)z_2^2 + 2Rz_2u + \tfrac{1}{2}R^2u^2 + p_1z_2 + p_2u \tag{25}$$

Now, from the minimum principle, we obtain two pairs of differential equations the state and adjoint variables must satisfy. First are the state equations, which we obtain from

$$\frac{\mathrm{d}z}{\mathrm{d}R} = \left(\frac{\partial\mathcal{H}}{\partial p}\right)' \tag{26}$$

Second are the adjoint equations,

$$\frac{\mathrm{d}p}{\mathrm{d}R} = -\left(\frac{\partial\mathcal{H}}{\partial z}\right)' = -\begin{bmatrix} 0 \\ p_1 + (4 + R^2\omega^2)z_2 + 2Ru \end{bmatrix} \tag{27}$$

The optimal control must minimize the Hamiltonian; hence, $(\partial\mathcal{H}/\partial u)(u) = 0$. We differentiate Eq. (25) to find the extremum of $\mathcal{H}$ with respect to u:

$$\frac{\partial\mathcal{H}}{\partial u} = 2Rz_2 + p_2 + R^2u = 0 \tag{28}$$

Solving for u,

$$u^* = -\left(\frac{2z_2}{R} + \frac{p_2}{R^2}\right) \tag{29}$$

where the superscript asterisk indicates an optimal quantity. To see that this control minimizes the Hamiltonian, check the second derivative of $\mathcal{H}$:

$$\frac{\partial^2\mathcal{H}}{\partial u^2} = R^2 > 0$$

Hence, Eq. (29) does minimize the Hamiltonian. Substituting Eq. (29) into the state and adjoint equations gives

$$\frac{\mathrm{d}}{\mathrm{d}R}\begin{bmatrix} z_1 \\ z_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -2z_2/R - p_2/R^2 \\ 0 \\ 4z_2 + 2p_2/R - p_1 - (4 + R^2\omega^2)z_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2/R & 0 & -1/R^2 \\ 0 & 0 & 0 & 0 \\ 0 & -R^2\omega^2 & -1 & 2/R \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ p_1 \\ p_2 \end{bmatrix} \tag{30}$$

Equation (30) is referred to as the Hamiltonian system associated with the optimal control problem. It has the form

$$\frac{\mathrm{d}}{\mathrm{d}R}\begin{bmatrix} z \\ p \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A' \end{bmatrix}\begin{bmatrix} z \\ p \end{bmatrix} \tag{31}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{2}{R} \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 \\ 0 & R^2\omega^2 \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{R^2} \end{bmatrix} \tag{32}$$

The solution to the linear homogeneous equation (31) is

$$\begin{bmatrix} z \\ p \end{bmatrix}(R_1) = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}(R_1, R_2)\begin{bmatrix} z \\ p \end{bmatrix}(R_2) \tag{33}$$

where the Ω_{ij} are unknown 2×2 matrices making up the state transition matrix and R_1, R_2 are specific times.

The known initial conditions for the state variables result in two boundary conditions. The other two boundary conditions come from transversality conditions. These pertain to various requirements imposed on the final angle of approach. We consider two cases in the following sections. First, we consider the case where the final angle of approach is free. In this case, the final angle of approach that minimizes the cost is automatically selected. Second, we consider a fixed final angle of approach. In this case, the final angle of approach is selected by the user based on operational considerations. The trajectory is optimized subject to this constraint.

B. Free Final Angle of Approach

For this case the transversality conditions give $p(0) = 0$. The initial state is $z(R_0) = (z_{10}, z_{20})$. Integrating Eq. (30) yields $p_1 = 0$. Substituting into Eq. (30) gives

$$\frac{\mathrm{d}p_2}{\mathrm{d}R} = -R^2\omega^2z_2 + \frac{2}{R}p_2 \tag{34}$$

Differentiating,

$$\frac{\mathrm{d}^2p_2}{\mathrm{d}R^2} = -R^2\omega^2\left(\frac{\mathrm{d}z_2}{\mathrm{d}R} + \frac{2}{R}z_2\right) + \frac{2}{R}\frac{\mathrm{d}p_2}{\mathrm{d}R} - \frac{2}{R^2}p_2 \tag{35}$$

Applying Eq. (30) and rearranging Eq. (35) yield the following equation:

$$\frac{\mathrm{d}^2p_2}{\mathrm{d}R^2} - \frac{2}{R}\frac{\mathrm{d}p_2}{\mathrm{d}R} + \left(\frac{2}{R^2} - \omega^2\right)p_2 = 0 \tag{36}$$

The general solution to Eq. (36) is

$$p_2(R) = C_1R\sinh(\omega R) + C_2R\cosh(\omega R) \tag{37}$$

Then z_2 can be solved for using Eq. (30):

$$\begin{aligned} z_2 &= \frac{(2/R)p_2 - (\mathrm{d}p_2/\mathrm{d}R)}{R^2\omega^2} \\ &= \frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] \\ &\quad + \frac{C_2}{R^2\omega^2}[\cosh(\omega R) - \omega R\sinh(\omega R)] \end{aligned} \tag{38}$$

To get the final solutions for p_2 and z_2 from Eqs. (37) and (38), we must evaluate C_1 and C_2 using the boundary constraints. The boundary condition $\boldsymbol{p}(0) = 0$ does not help out. However, if the missile is to hit the target, the heading error angle at $R = 0$ must be zero [$\delta(0) = 0$]. From physical considerations, $\delta(R) = x_2(R)$ is continuous, so the heading error angle should approach zero as R approaches zero; i.e.,

$$\lim_{R\to 0} x_2(R) = 0 \tag{39}$$

(Only continuous solutions for the state and adjoint variables are allowed.) When $C_1 = 0$ we have, from Eq. (38),

$$\begin{aligned} z_2(R) &= \frac{C_2}{R^2\omega^2}[\cosh(\omega R) - \omega R \sinh(\omega R)] \\ &= \frac{C_2}{R^2\omega^2}\left\{\left[1 + \frac{1}{2}\omega^2 R^2 + \frac{1}{24}\omega^2 R^4 + \mathcal{O}(R^6)\right]\right. \\ &\quad \left. - \left[\omega^2 R^2 + \frac{1}{6}\omega^4 R^4 + \mathcal{O}(R^6)\right]\right\} \\ &= \frac{C_2}{R^2\omega^2}\left[1 - \frac{1}{2}\omega^2 R^2 - \frac{1}{8}\omega^4 R^4 + \mathcal{O}(R^6)\right] \end{aligned} \tag{40}$$

Now, Eq. (7) gives

$$\begin{aligned} x_2(R) &= -\tan^{-1}(Rz_2) = -\left[Rz_2 - \frac{1}{3}(Rz_2)^3 + \mathcal{O}\big((Rz_2)^5\big)\right] \\ &= -\frac{C_2}{8R}\frac{[-8 + 4(\omega R)^2 + (\omega R)^4]}{\omega^2} \\ &\quad + \frac{C_2^3}{1536R^3}\frac{[-8 + 4(\omega R)^2 + (\omega R)^4]^3}{\omega^6} + \text{HOT} \end{aligned}$$

(Here, HOT stands for higher order terms.) This shows $x_2(R)$ goes to infinity as $R \to 0$ unless $C_2 = 0$. Hence, to satisfy Eq. (39), it is necessary that $C_2 = 0$. When $C_2 = 0$, we have, from Eq. (38),

$$\begin{aligned} z_2(R) &= \frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R \cosh(\omega R)] \\ &= \frac{C_1}{R^2\omega^2}\left\{\left[\omega R + \frac{1}{6}\omega^3 R^3 + \mathcal{O}(R^5)\right]\right. \\ &\quad \left. - \left[\omega R + \frac{1}{2}\omega^3 R^3 + \mathcal{O}(R^5)\right]\right\} \\ &= -\frac{1}{3}C_1\omega R + \text{HOT} \end{aligned}$$

$$x_2(R) = -\tan^{-1}(Rz_2) = -\frac{1}{3}C_1\omega R^2 + \text{HOT}$$

In this case Eq. (39) is satisfied for all $0 < C_1 < \infty$.

Note that the terms in the series expansion of x_2 associated with C_1 are even functions of R and the terms associated with C_2 are odd functions of R. This fact implies that we may do an analysis by taking $C_1 = 0$ and $C_2 \neq 0$ and then $C_1 \neq 0$ and $C_2 = 0$. With $C_2 = 0$, we get

$$z_2(R) = \frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R \cosh(\omega R)] \tag{41}$$

$$p_2(R) = C_1 R \sinh(\omega R) \tag{42}$$

From Eqs. (30) and (41)

$$z_1(R) = -C_1\frac{\sinh(\omega R)}{R\omega^2} + K_2 \tag{43}$$

The integration constants C_1 and K_2 may be computed from the initial values of γ and δ. From Eqs. (6) and (41) we have

$$-\frac{\tan[\delta(R_0)]}{R_0} = \frac{C_1}{R_0^2\omega^2}[\sinh(\omega R_0) - \omega R_0 \cosh(\omega R_0)]$$

Solving this equation in C_1 gives

$$C_1 = \frac{-R_0\omega^2 \tan[\delta(R_0)]}{\sinh(\omega R_0) - \omega R_0 \cosh(\omega R_0)} \tag{44}$$

From Eqs. (6) and (43) we have

$$-\gamma_0 + \delta_0 = -C_1\frac{\sinh(\omega R_0)}{R_0\omega^2} + K_2$$

Hence,

$$K_2 = \frac{C_1 \sinh(\omega R_0)}{R_0^2\omega^2} + \delta_0 - \gamma_0 \tag{45}$$

Next, closed-form solutions for the (untransformed) state variables are given. From Eqs. (7), (41), and (43), we get

$$\begin{aligned} \gamma(R) &= -\left(-C_1\frac{\sinh(\omega R)}{R\omega^2} + K_2\right) \\ &\quad - \tan^{-1}\left(R\frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R \cosh(\omega R)]\right) \end{aligned} \tag{46}$$

$$\delta(R) = -\tan^{-1}\left(R\frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R \cosh(\omega R)]\right) \tag{47}$$

Such a closed-form solution for the state variables given by Eqs. (46), (47), (44), and (45) was not obtained in the original solution method for this problem. It is reiterated that the closed-form solution obtained here is an approximate solution to the optimal control problem given by Eqs. (1) and (2) because of the quadratic approximation made in the cost function in Eq. (19). (No other approximation is made.)

C. Intercept Angle Terminal Constraint

In this section, the LQR optimal control problem is formulated with a terminal constraint on the intercept angle. The terminal constraint is $x_1 = \gamma_f$. In transformed coordinates [see Eq. (6)], using the continuity of the state variables, the terminal constraint is

$$\lim_{R\to 0}\left[-z_1 - \tan^{-1}(Rz_2)\right] = \gamma_f \tag{48}$$

The two initial boundary conditions for Eq. (30) are

$$z(R_0) = (z_{10}, z_{20}) \tag{49}$$

An integration of Eq. (30) gives

$$p_1 = \text{const} = K_1 \tag{50}$$

where K_1 is to be determined. (Integration constants in this section are not to be confused with integration constants of the previous section.) The remaining state and adjoint equations are

$$\frac{dz_1}{dR} = z_2 \tag{51}$$

$$\frac{dz_2}{dR} = -2\frac{z_2}{R} - \frac{p_2}{R^2} \tag{52}$$

$$\frac{dp_2}{dR} = -K_1 - R^2\omega^2 z_2 + \frac{2p_2}{R} \tag{53}$$

Here, p_2 has the same general solution as before:

$$p_2(R) = C_1 R \sinh(\omega R) + C_2 R \cosh(\omega R) \tag{54}$$

Applying Eqs. (53) and (54),

$$z_2(R) = \frac{2(p_2/R) - (\mathrm{d}p_2/\mathrm{d}R) - K_1}{R^2\omega^2}$$

$$= \frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] + \frac{C_2}{R^2\omega^2}[\cosh(\omega R) - \omega R\sinh(\omega R)] - \frac{K_1}{R^2\omega^2} \quad (55)$$

The same situation as before applies and is used to evaluate the coefficients. Considering the Taylor series expansion of x_2 using Eqs. (55) and (6), the term $K_1/R^2\omega^2$ will contribute terms with even powers of R. Referring to Eq. (40), the Taylor series expansion of z_2 when $C_1 = 0$ is

$$z_2(R) = \frac{C_2}{R^2\omega^2}\left[1 - \frac{1}{2}\omega^2R^2 - \frac{1}{8}\omega^4R^4 + O(R^6)\right] + \frac{K_1}{R^2\omega^2}$$

From this, it can be seen that a requirement for satisfying Eq. (39) is that $K_1 = C_2$. Therefore,

$$z_2(R) = \frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] + \frac{C_2}{R^2\omega^2}[\cosh(\omega R) - \omega R\sinh(\omega R) - 1] \quad (56)$$

From Eqs. (51) and (56), we get

$$z_1(R) = -\frac{1}{R\omega^2}[C_1\sinh(\omega R) + C_2\cosh(\omega R) - C_2] + K_2 \quad (57)$$

where K_2 is an integration constant. Applying the terminal constraint (48), we get

$$K_2 = \frac{C_1}{\omega} - \gamma_f \quad (58)$$

Hence

$$z_1(R) = -\frac{1}{R\omega^2}[C_1\sinh(\omega R) + C_2\cosh(\omega R) - C_2] + \frac{C_1}{\omega} - \gamma_f \quad (59)$$

The constants C_1 and C_2 can be determined from the initial state $x(R_0)$. From Eqs. (55) and (59) we obtain

$$z_1(R_0) = -\gamma_0 + \delta_0 = -\frac{1}{R_0\omega^2}[C_1\sinh(\omega R_0) + C_2\cosh(\omega R_0) - C_2] + \frac{C_1}{\omega} - \gamma_f$$

$$z_2(R_0) = -\frac{\tan(\delta_0)}{R_0} = \frac{C_1}{R_0^2\omega^2}[\sinh(\omega R_0) - \omega R_0\cosh(\omega R_0)] + \frac{C_2}{R_0^2\omega^2}[\cosh(\omega R_0) - \omega R_0\sinh(\omega R_0) - 1]$$

Hence, C_1 and C_2 are the solution of the linear algebraic equation:

$$\begin{bmatrix} R_0\omega - \sinh(\omega R_0) & 1 - \cosh(\omega R_0) \\ \begin{pmatrix} \sinh(\omega R_0) \\ -\omega R_0\cosh(\omega R_0) \end{pmatrix} & \begin{pmatrix} \cosh(\omega R_0) - 1 \\ -\omega R_0\sinh(\omega R_0) \end{pmatrix} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = R_0\omega^2 \begin{bmatrix} \gamma_f - \gamma_0 + \delta_0 \\ -\tan(\delta_0) \end{bmatrix} \quad (60)$$

At this point, the closed-form solutions to the (untransformed) state variables for the constrained final angle of approach case may be obtained. We have

$$\gamma(R) = -\frac{1}{R\omega^2}[C_1\sinh(\omega R) + C_2\cosh(\omega R) - C_2] + K_2 - \tan^{-1}\left(\frac{C_1}{R\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] + \frac{C_2}{R\omega^2}[\cosh(\omega R) - \omega R\sinh(\omega R) - 1]\right) \quad (61)$$

$$\delta(R) = -\tan^{-1}\left(\frac{C_1}{R\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] + \frac{C_2}{R\omega^2}[\cosh(\omega R) - \omega R\sinh(\omega R) - 1]\right) \quad (62)$$

We have arrived at closed-form solutions for the state variables for the constrained intercept angle case. These are given by Eqs. (61) and (62). Such closed-form solutions were not obtained in the original formulation. See Ref. 1.

VI. Numerical Examples

In this section, several numerical examples of the linearized kappa guidance are given. A comparison to the original method, which is presented in Lin,[1] is given. These results are shown in Figs. 2–4. Each figure shows a trajectory generated by the original kappa method and a trajectory generated by the linearized kappa method. In each case, the origin of the trajectory (launch point of the missile) is at (0, 0) and the predicted intercept point (PIP) is at (20, 0), giving $R_0 = 20$. The constant ω is set to 1. For all of the numerical examples, the program was set to stop when $R \leq 0.15$. The abscissa on these figures is downrange and the ordinate is altitude above a

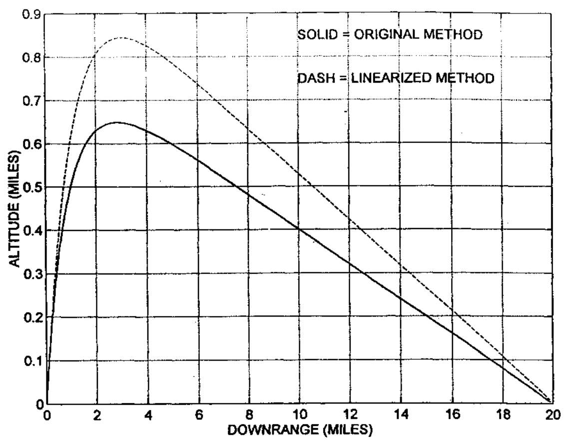

Fig. 2 $\gamma_0 = \frac{1}{4}\pi$, γ_f = free.

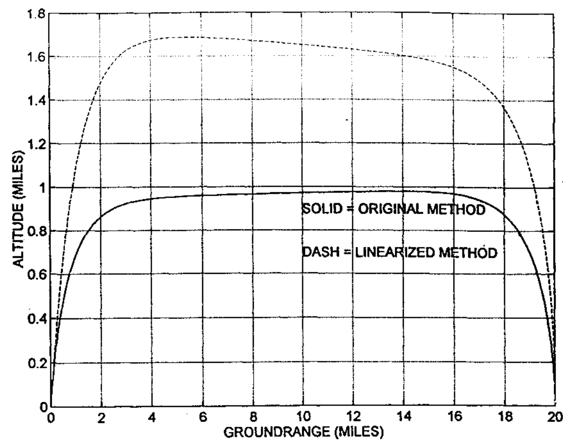

Fig. 3 $\gamma_0 = \frac{1}{3}\pi$, $\gamma_f = -\frac{1}{2}\pi$.

Table 1 Cost original method vs linearized method

γ_0, δ_0	γ_f	Cost Original	Cost Linearized
$\frac{1}{4}\pi$	Free	20.21	20.22
$\frac{1}{4}\pi$	$-\frac{1}{4}\pi$	20.41	20.47
$\frac{1}{3}\pi$	$-\frac{1}{3}\pi$	20.80	21.06
$\frac{1}{3}\pi$	$-\frac{1}{2}\pi$	21.14	21.64
$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	21.14	21.76

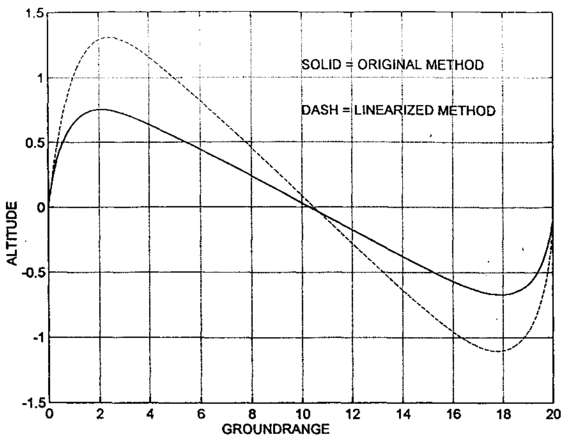

Fig. 4 $\gamma_0 = \frac{1}{3}\pi, \gamma_f = \frac{1}{2}\pi$.

reference launch altitude h_{ref}. (Plots depict h_{ref} vs. range.) No particular units are represented; however, these figures may be considered to be in miles. In Fig. 2, $\gamma_0 = \delta_0 = \frac{1}{4}\pi$ and $\gamma_f =$ free. In Figs. 3 and 4, $\gamma_0 = \delta_0 = \frac{1}{3}\pi$. In Fig. 3, $\gamma_f = -\frac{1}{2}\pi$. In Fig. 4, $\gamma_f = \frac{1}{2}\pi$. The trajectory in Fig. 4 is presented for the purpose of demonstrating a terminal approach opposite that of Fig. 3 and to provide a more complete comparison of the linearized kappa guidance method with the original kappa guidance method.

The costs for all trajectories are computed with the same cost function, given by Eq. (2). The costs are tabulated in Table 1, along with some examples whose results are not plotted. We see from these results that although the cost for the linearized method is slightly greater than the original solution, it is comparable. We also see from Table 1 that heading error angles of $\frac{1}{3}\pi$ are used in the linearized method and resulted in successful trajectories. This is intended to illustrate the limits of the quadratic approximation used in Sec. IV. (A linear approximation, such as $\sin\theta \approx \theta$, as a general rule of thumb, requires $\theta < \frac{1}{12}\pi$.) Neither method generates a successful trajectory for $\gamma_0 = \delta_0 = \frac{1}{2}\pi$. Presumably $\delta_0 = \frac{1}{2}\pi$ is too great a heading error angle for either method to work.

As was mentioned previously, both the original solution and the linearized solution are suboptimal solutions that make different approximations. Since these two methods generate two different suboptimal solutions, we do not expect that the trajectories in the figures match up. The trajectories in these figures were computed differently for each method. Considering the linearized kappa guidance method, first an incremental length along the trajectory Δs is set. The current values of the state variables, the downrange and altitude coordinates, and the range R to the PIP are assumed given. These variables are substituted into Eqs. (46) and (47) for the free terminal constraint case and into Eqs. (61) and (62) for the fixed final angle of approach case. These formulas give the new velocity vector angle γ the missile should have. Using γ and Δs, the trajectory is incremented. What has been described is an Euler method, whereas what is actually used is a two-point method. The velocity vector angle used is the average of that computed at the beginning and end of a trajectory increment. For the original method, again the current values of the state variables, the downrange and altitude coordinates and the range R to the PIP are assumed given. Then, the optimal input κ to the plant equations (1) is computed using the optimal feedback controller equations previously mentioned in Lin[1] and the plant is integrated. The downrange and altitude coordinates are incremented the same as with the linearized method.

If the PIP were to change at some point during the generation of the trajectory, the new PIP would be used in the computation of the next trajectory increment for either method, and the trajectory computation would proceed as described above. Now suppose a missile is flying a midcourse trajectory with free final angle of approach. Let the PIP move sometime during the course of the flight in such a way that the final angle of approach is into the sun (i.e., undesirable in some way). With the linearized method discussed in this report, such a circumstance could be checked using the closed-form solution for the velocity vector angle given by Eq. (46), and if necessary the rest of the midcourse trajectory could be recomputed using a fixed final angle of approach specified to be an angle, e.g., not into the sun. With the original method, such a circumstance could be checked only by computing the whole trajectory.

VII. Conclusion

This paper presents an alternate solution to the kappa guidance optimal control problem. The primary feature of this alternate solution is that closed-form solutions to the state variables have been computed for both the free terminal angle of approach case and the fixed terminal angle of approach case. Such solutions do not exist for the original method. The closed-form solutions to the state variables for the linearized method established in this report may be advantageous in certain operational or simulation situations. For example, simulation studies may require less computing through the use of the closed-form solutions. In operational situations requirements may be presented, for example, that certain velocity vector angles be avoided. The closed-form solutions may be able to quickly determine if such requirements are met. Finally, we remark that the technique of linearization with quadratic cost approximation may be of use as a general technique for computing suboptimal solutions to optimal control problems.

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